JMC 2015 Teacher's notes

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Competition	Year	Question	Skills	Problem-solving
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Competition	Year	Question	Skills	Problem-solving
JMC	2015	1	• Number / Adding and subtracting integers • Number / Negative numbers	• Elimination • Deduction • Alternative strategies

1. Which of the following calculations gives the largest answer?

A $1-2+3+4$ B $1+2-3+4$ C $1+2+3-4$ D $1+2-3-4$ E $1-2-3+4$

Teacher's note:
The main interest of this question is that, because the sum are simple, students might be tempted to do all of them and compare. They will find A is the correct answer.
But because of its simplicity, this makes it the ideal sort of question to stimulate young beginners in problem-solving to find more efficient and conceptually superior strategies. They may come up with 2 kinds of strategies:

- Narrowing down the possibilities: the 2 biggest numbers in each sum are 3 and 4. That rules out D and C; etc. before finding out that A is the biggest. This strategy is not the most rigorous and optimal, but it should be praised because it's already infinitely better than just calculating the 5 sums.
- Spotting the key point, i.e. the 4 numbers in each sum are the same, therefore you only need to compare how much is subtracted.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	2	• Measuring / Time units	

2. It has just turned 22:22. How many minutes are there until midnight?

A 178 B 138 C 128 D 108 E 98

Teacher's note:
Straightforward, as kids will naturally partition the time gap (38 + 60).
Nice extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	3	• Number / Estimating • Number / Properties of number	• Elimination • Alternative strategies

3. What is the value of $\dfrac{12\,345}{1+2+3+4+5}$?

A 1 B 8 C 678 D 823 E 12 359

Teacher's note:
Once the denominator has been calculated (15), there are 3 possible strategies:

- Work out the answer to the division 12345 / 15; but, although a long division can be avoided by partitioning the numerator, it's not the quickest way and therefore students should be pushed further.
- 1^{st} elimination strategy: as explained in UKMT solutions, odd divided by odd is even, which eliminates B and C; can't be 1, can't be 12359, therefore 823 is the right answer.
- 2^{nd} elimination strategy (more direct, and more in the 'feel for numbers' line of thinking): if denominator was 15000 instead of 15, the result would be 'a bit less than 1000', so only C and D are possible. As 12/15 is 4/5 or 0.8 or 80%, 823 has to be right.

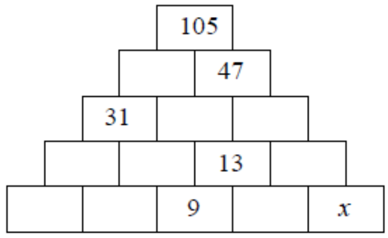
Interesting extensions in UKMT solutions, using divisibility criteria.
Other possible extension: work out the formula for the sum of consecutive integers.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	4	• Number / Adding and subtracting integers • Algebra / Forming and solving linear equations	• Deduction • Economy

4. In this partly completed pyramid, each rectangle is to be filled with the sum of the two numbers in the rectangles immediately below it.

What number should replace x?

A 3 B 4 C 5 D 7 E 12



Teacher's note:
This is a straightforward number pyramid, but there are 2 interesting problem-solving features:

- Deduction strategy: filling in pyramid from the top
- Economy:
  - No need to write equations, numbers can be worked out directly from top to bottom using subtraction.
  - Understanding that, because x is your target, you don't need to find the 4 bottom left numbers

Possible extension (algebra): as in UKMT solutions, name all missing numbers (p,q,r,s,t), write the equations and solve.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	5	• Fractions / Adding and subtracting fractions • Algebra / Forming and solving linear equations	• Name / Label • Bar modelling

5. The difference between $\frac{1}{3}$ of a certain number and $\frac{1}{4}$ of the same number is 3. What is that number?

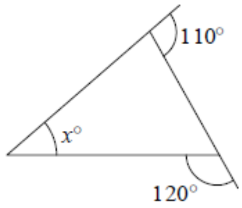
A 24 B 36 C 48 D 60 E 72

Teacher's note:
This is a straightforward subtraction of fractions, with 12 as common denominator.
However, for students who do not 'jump conceptually' at this solution, bar modelling is an ideal visual tool, as often with arithmetic problems. In the end, students will still have to write the same operation (subtraction of 2 fractions) and find the common denominator, but drawing 2 bars of the same number and observing that the difference in both divisions equals 3 provides a useful visual interpretation of the problem.
Also possible to name the missing number x and write the equation. Which could be a stepstone towards the second extension in UKMT solutions (writing a formula with 4 variables).
In any case, it does make it easier for students to actually name the number they're trying to find.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	6	• Geometry / Angles in a triangle	• Deduction

6. What is the value of x in this triangle?

A 45 B 50 C 55 D 60 E 65



Teacher's note:
UKMT solutions suggests using the External Angle Theorem, which is quite elegant and can be a preliminary to the extension (proving the theorem).
I would favour a 'comfort zone' solution that only uses interior angles only and a quick deduction process: (1) find 60° and 70° from 120° and 110°; which, out of 180° (sum of interior angles), leaves 50°.
Extension (algebra): write the corresponding equations and solve.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	7	• Numbers / Factors and multiples	• Economy • Deduction • Alternative strategies

7. The result of the calculation 123 456 789 × 8 is almost the same as 987 654 321 except that two of the digits are in a different order. What is the sum of these two digits?

A 3 B 7 C 9 D 15 E 17

Teacher's note:
This is a very nice problem to drive students towards economy and show them you can 'do a lot with very little':

- Working out the multiplication is acceptable, but slow, therefore other strategies should be found.
- Starting the multiplication and stop and the first 3 digits is already better.
- Noticing that 8x9 is 72 and deducing that 2 and 1 must be interchanged digits is best.

Nice extensions in UKMT solutions on divisibility criteria.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	8	• Numbers / Factors and multiples	• Elimination • Deduction • Alternative strategies • Easy way out

8. Which of the following has the same remainder when it is divided by 2 as when it is divided by 3?

A 3 B 5 C 7 D 9 E 11

Teacher's note:

It is possible to try all the answers until you find one number that works, i.e. 7. However, students can be pushed further to find at least one of these 2 strategies:

- Two of these numbers (3 and 9) are multiples of 3, but they are not multiples of 2, so their remainder when divided by 2 and 3 cannot be the same. That leaves out only 3 solutions to try.
- Easy way out: as pointed out in UKMT solutions, all proposed numbers are odd, therefore the 'same remainder' is not a mystery anymore: it has to be 1 (non-exact division by 2). Among the proposed numbers, only 7 has remainder 1 when divided by 3.

Nice extensions in UKMT solutions on various properties of numbers.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	9	• Number / Estimating	• Bar modelling

9. According to a newspaper report, "A 63-year-old man has rowed around the world without leaving his living room." He clocked up 25 048 miles on a rowing machine that he received for his 50th birthday.

Roughly how many miles per year has he rowed since he was given the machine?

A 200 B 500 C 1000 D 2000 E 4000

Teacher's note:

Straightforward, once students find the man has been rowing for 13 years. 25 is close to 26, which divides well by 13.

There again, for learning purposes, bar modelling can be used to represent the man's age as one bar with 2 divisions (50 and 13), the last division representing 25048 miles.

Nice extensions in UKMT solutions on range of possible values.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	10	• Number / 4 operations on integers	

10. In the expression 1 ☐ 2 ☐ 3 ☐ 4 each ☐ is to be replaced by either + or ×.

What is the largest value of all the expressions that can be obtained in this way?

A 10 B 14 C 15 D 24 E 25

Teacher's note:

The key point here is to 'resist the temptation' to multiply everything, as multiplying by 1 doesn't increase the overall value.

Nice extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	11	• Number / Properties of numbers	• Systematic list

11. What is the smallest prime number that is the sum of three different prime numbers?

A 11 B 15 C 17 D 19 E 23

Teacher's note:
This question is a great opportunity for students to learn a couple of tricks:

- Typically with questions about prime numbers, squares or cubes, it's quite handy to make a quick list on rough paper, it makes it more visual.
- As soon as you know the question is about prime numbers, always bear in mind that:
  - 1 is not a prime number
  - The only even prime number is 2

Using the list will quickly lead to the solution (19).
However, knowing that 2 is the only even prime number quickens up the process, because the sum of 2 + two other prime numbers is even, and therefore not prime. So the smallest possible number of the sum has to be 3.
Similar extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	12	• Fractions / Adding and subtracting fractions • Algebra / Forming and solving linear equations	• Name / Label • Bar modelling

12. A fish weighs the total of 2 kg plus a third of its own weight. What is the weight of the fish in kg?

A $2\frac{1}{3}$ B 3 C 4 D 6 E 8

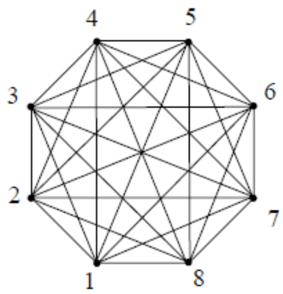
Teacher's note:
As for question 5, there is the choice between using fractions (2 kg is 2 thirds of the weight) or algebra (write the equation and solve for weight). For students who experience some difficulty in conceptualizing this kind of arithmetic problem, bar modelling is a wise choice.
If processed conceptually or through algebra, it makes it easier for students to actually name the number they're trying to find.
Similar extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	13	• Number / Adding and subtracting integers • Number / Factors and multiples	• Systematic list

13. In the figure shown, each line joining two numbers is to be labelled with the sum of the two numbers that are at its end points.

How many of these labels are multiples of 3?

A 10 B 9 C 8 D 7 E 6



Teacher's note:
This is a nice question to illustrate the 'Systematic list' strategy. As explained in the UKMT solution, the first limiting factor is the maximum sum (7+8=15), which limits the list of multiples of 3 (3, 6, 9, 12, 15).
Then, there's a 'list inside the list', as it's just a matter of listing number bonds for each multiple of 3.
Similar extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	14	• Number / Properties of numbers	• Elimination

14. Digits on a calculator are represented by a number of horizontal and vertical illuminated bars. The digits and the bars which represent them are shown in the diagram.

How many digits are both prime and represented by a prime number of illuminated bars?

A 0 B 1 C 2 D 3 E 4

Teacher's note:

This is typically an elimination problem, as only 4 digits are prime. Straightforward then, as students just need to count illuminated bars for each of the 4 prime digits.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	15	• Number / Factors and multiples	• Elimination • Alternative strategies

15. Which of the following is divisible by all of the integers from 1 to 10 inclusive?

A 23×34 B 34×45 C 45×56 D 56×67 E 67×78

Teacher's note:

There again, typically an elimination problem, and therefore the key to finding a quick solution is to use the most restrictive divisibility criteria. Several possibilities:

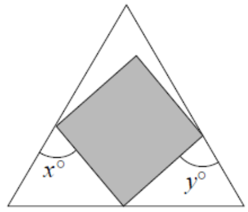
- Divisibility by 5 is a pretty efficient choice, because it rules out A, D and E right away. Then, in order to choose between B and C, divisibility by 4 is quite direct because, 45 being odd, students just have to deal with 34 and 56, therefore C is the only one that works.
- UKMT solution uses divisibility by 3, 7 and 5, which works as well, but slower.

Similar extensions in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	16	• Geometry / Polygons / Angles in a triangle • Algebra / simultaneous equations • Algebra / Rearranging formulae	• Name / Label • Alternative strategies

16. The diagram shows a square inside an equilateral triangle.
What is the value of $x + y$?

A 105 B 120 C 135 D 150 E 165



Teacher's note:
The most natural solution is Method 1 as described in UKMT solutions, which works it out using angles around point R (where the square meets the base of the triangle).
There is a case for naming angles or not naming angles, that is really a matter of point of view. What should be our position as teachers ? On the one hand, not naming angles does make it a bit difficult because you have to do it 'all in your head'. On the other hand, naming angles means entering a process of writing simultaneous equations and solving them (without trying to solve for individual variables, therefore sticking to p+q and x+y), which requires a bit more algebra proficiency from students.
Conclusion: your position as a teacher will simply depend on which students you are teaching, where they are, and what is your purpose in giving them this problem to solve. For example, the question could be used to introduce students with a smart way of using simultaneous equations (or rearranging formulae, as may be the case).

UKMT solution proposes 2 other methods. As explained, Method 2 is ingenious but lacks conceptual generality. Method 3 is very geometric and, not being a great fan of using external angles, I would probably skip that. There is, however, another method that I would definitely ask my students to find, using the top point of the square (let's call it V).Here it is:

- The sum of the internal angles of the quadrilateral PUTV is 360°, therefore <UPV + <UTV = 30°;
- As <VPR = <VTR = 90°, then x + y = 2x180° - 2x90° - 30° = 150°.

Come to think of it, wouldn't that be the easiest method ?

Competition	Year	Question	Skills	Problem-solving
JMC	2015	17	• Binary logic	• Tree • Economy

17.

Knave of Hearts: "I stole the tarts."
Knave of Clubs: "The Knave of Hearts is lying."
Knave of Diamonds: "The Knave of Clubs is lying."
Knave of Spades: "The Knave of Diamonds is lying."

How many of the four Knaves were telling the truth?

A 1 B 2 C 3 D 4 E more information needed

Teacher's note:

This kind of question belongs to a long-time tradition of riddles about liars and assorted characters.Each of them represents 2 possibilities (true or false) and whether what they are saying is true or not has implications as to whether some of the other characters are liars or not.

Typically, these questions are best solved using a logical Tree. In this instance, it's more efficient to start with the last character (Spades) and work out all the logical threads from there, which goes like this:

- Thread 1: Spades true -> Diamonds lying -> Clubs true -> Hearts lying -> Hearts did not steel the tarts
- Thread 2: Spades lying -> Diamonds true -> Clubs lying -> Hearts true -> Hearts stole the tarts

Logical Trees are really handy and visual when done on rough paper, and it also helps to discuss the solution with others.

There is a bit of Economy involved as well, as it is not necessary to pursue each thread until its conclusion as to who stole the tart. In a way, Economy is not fundamental in this problem, but it does help students to remain focused on the question: this is about how many knaves are telling the truth, NOT about who stole the tart…

Which could be an extension question, for instance: 'in the end, do we know who stole tarts ?' The answer is no, because each of the 2 threads is equally likely and leads to opposite conclusions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	18	• Fractions / Equivalent fractions	• Easy way out

18. Each of the fractions $\frac{2637}{18\,459}$ and $\frac{5274}{36\,918}$ uses the digits 1 to 9 exactly once. The first fraction simplifies to $\frac{1}{7}$. What is the simplified form of the second fraction?

A $\frac{1}{8}$ B $\frac{1}{7}$ C $\frac{5}{34}$ D $\frac{9}{61}$ E $\frac{2}{7}$

Teacher's note:

This is the kind of question where observation provides a shortcut to the solution (easy way out). In this instance, students would be expected to notice that 5274 is twice 2637 and 36918 is twice 18459, therefore both fractions are equivalent.

All other solutions (divisibility criteria, unit digit, etc.) are acceptable but much slower.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	19	• Number / Properties of numbers	• Systematic list • Elimination

19. One of the following cubes is the smallest cube that can be written as the sum of three positive cubes. Which is it?

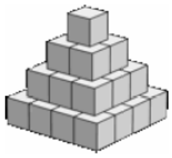
A 27 B 64 C 125 D 216 E 512

Teacher's note:
Step 1 is definitely a list. It is useful for students to be familiar with the most common systematic lists: list of first squares, list of first cubes, list of first prime numbers, etc.
Step 2 is elimination of solutions A to C, for numerical reasons.
Nice extensions in UKMT solutions., including beautiful references to Maths icons Ramanujan and Fermat.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	20	• Measuring / Area and Volume • Algebra / Sequences	

20. The diagram shows a pyramid made up of 30 cubes, each measuring $1\,\text{m} \times 1\,\text{m} \times 1\,\text{m}$.
What is the total surface area of the whole pyramid (including its base)?

A $30\ \text{m}^2$ B $62\ \text{m}^2$ C $72\ \text{m}^2$ D $152\,\text{m}^2$ E $180\,\text{m}^2$



Teacher's note:
The method from UKMT solutions is quite elegant. It's also quite all right to proceed from the bottom layer to the next one, adding up areas of the base (twice) and sides, then subtracting the base of the next layer, etc.
The sequence-related extension is very tempting.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	21	• Geometry / 2D shapes • Algebra / Forming and solving linear equations	• Name / Label

21. Gill is now 27 and has moved into a new flat. She has four pictures to hang in a horizontal row on a wall which is 4800 mm wide. The pictures are identical in size and are 420 mm wide. Gill hangs the first two pictures so that one is on the extreme left of the wall and one is on the extreme right of the wall. She wants to hang the remaining two pictures so that all four pictures are equally spaced. How far should Gill place the centre of each of the two remaining pictures from a vertical line down the centre of the wall?

A 210 mm B 520 mm C 730 mm D 840 mm E 1040 mm

Teacher's note:
This is a nice DIY geometry problem. Best way is probably to sketch a rough diagram on paper and name the gap (g) between each picture, then solve for g. Noticing the symmetry means you can focus on one half of the diagram, but that isn't really a time-saver.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	22	• Geometry / 2D shapes • Geometry / Symmetry • Fractions / Introducing fractions	• Complete the grid • Economy

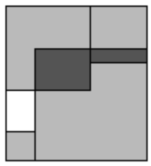
22. The diagram shows a shaded region inside a regular hexagon. The shaded region is divided into equilateral triangles. What fraction of the area of the hexagon is shaded?

A $\frac{3}{8}$ B $\frac{2}{5}$ C $\frac{3}{7}$ D $\frac{5}{12}$ E $\frac{1}{2}$

Teacher's note:
Completing the grid is an essential step for this kind of question (regular hexagon, equilateral triangles), but it is the sixfold symmetry that provides a real economy, as it is possible to focus on just one corner of the hexagon.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	23	• Measuring / Area	• Deduction

23. The diagram shows four shaded glass squares, with areas 1 cm^2, 4 cm^2, 9 cm^2 and 16 cm^2, placed in the corners of a rectangle. The largest square overlaps two others. The area of the region inside the rectangle but not covered by any square (shown unshaded) is 1.5 cm^2.
What is the area of the region where squares overlap (shown dark grey)?



A $2.5 cm^2$ B $3 cm^2$ C $3.5 cm^2$ D $4 cm^2$ E $4.5 cm^2$

Teacher's note:
This is a a series of deductions as each measurement leads to another around the rectangle.
Method 2 outlined in UKMT solutions, which is based on areas only, is smart, but a lot more difficult for students.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	24	• Number / Place value • Number / Factors and multiples	• Deduction

24. A *palindromic number* is a number that reads the same when the order of its digits is reversed. What is the difference between the largest and smallest five-digit palindromic numbers that are both multiples of 45?

A 9180 B 9090 C 9000 D 8910 E 8190

Teacher's note:

This is a beautiful question, and a highly educational one too, because it forces students to process each piece of information very carefully and translate that into a series of mathematical deductions. as each measurement leads to another around the rectangle.

Interpreting each information provides one more piece of the jigsaw puzzle:

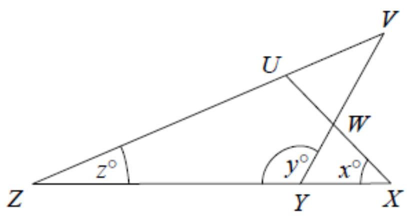
- Five-digit palindromic number -> Can be written 'abcba' with a>0
- Multiple of 45 -> Mulltiple of 5 -> last digit is 5 or 0 -> can't be 0 because last digit = first digit -> has to be 5 -> digits add up to 10 + 2b + c
- Multiple of 9 -> sum of digits is a multiple of 9 -> 2b + c has remainder 1 when divided by 9

Then it's just a matter of trying out solutions with lowest possible hundreds (0) and highest possible hundreds (9).

Nice extensions to related topics in UKMT solutions.

Competition	Year	Question	Skills	Problem-solving
JMC	2015	25	• Geometry / Angles in a triangle • Algebra / Forming equations • Algebra / Rearranging formulae	• Deduction

25. The four straight lines in the diagram are such that $VU = VW$. The sizes of $\angle UXZ$, $\angle VYZ$ and $\angle VZX$ are $x°$, $y°$ and $z°$. Which of the following equations gives x in terms of y and z?



A $x = y - z$ B $x = 180 - y - z$ C $x = y - \frac{z}{2}$

D $x = y + z - 90$ E $x = \frac{y - z}{2}$

Teacher's note:

Although set in a geometrical context, this question is first and foremost a very good exercise on combining and rearranging 3-variable formulae without getting lost.

The initial deductive steps are from geometry, starting with angles in an isosceles triangle and opposite angles.